THE UNIVERSITY OF WESTERN ONTARIO
FACULTY OF ENGINEERING SCIENCE
DEPARTMENT OF ELECTRICAL ENGINEERING

E.S. 430b COMPUTATIONAL ELECTROMAGNETICS

Final Examination - April 27, 1993, 2:00 P.M.

Time allotted: Three Hours

General Instructions:

1) This is an **open book** exam.

2) Calculators **are** permitted, but **sharing** of calculators between students is **not** permitted.

3) There are **7** questions in total.

4) Answer **all** questions in the booklets provided; redraw figures and tables in your booklet when it is necessary.

5) The marks allotted for each question are indicated in the left margin next to each question number.

6) Make sure that your name and student number are written at the front of each booklet you use and number the booklets if you use more than one.

7) Print clearly.

8) Clearly indicate the steps taken in your answers as part marks will be given.

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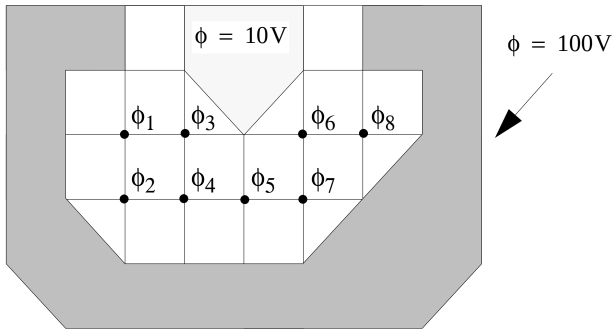
10 1) For the two dimensional problem shown in the figure, determine the matrix equation which would result from applying a centered difference second order approximation to solve Laplace's equation at each grid point shown. Number your solution vector according to the numbering shown in the figure.



10 2) (a) For the matrix equation of question 1, derive the update equations of the Successive Over-relaxation iterative scheme, using an over-relaxation constant of $\omega = 1.5$, for example:

$$\phi_1^{(n+1)} = 1.5\left[\frac{1}{4}\left(\phi_2^{(n)} + \phi_3^{(n)} + 200\right) - \phi_1^{(n)}\right] + \phi_1^{(n)} .$$

(b) Use the initial guess shown in table 1 to determine the values after the first iteration as well as the relative displacement norm in the final column.

Table 1: SOR, $\omega = 1.5$

iteration number	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	$\frac{\Vert\Delta\phi\Vert}{\Vert\phi\Vert}$
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0
1									

20 3) The loss-less transmission line equations can be written in matrix form in terms of the voltage, $V(x, t)$, and the current, $I(x, t)$, along the line as:

$$\frac{\partial}{\partial t}\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} + \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where L is the distributed inductance [H/m], and C is the distributed capacitance [F/m]. Consider the grid functions given as:

$$V(i\Delta x, n\Delta t) = V_i^n \qquad I(i\Delta x, n\Delta t) = I_i^n \qquad \mathbf{u}_i^n = \begin{bmatrix} V_i^n & I_i^n \end{bmatrix}^T$$

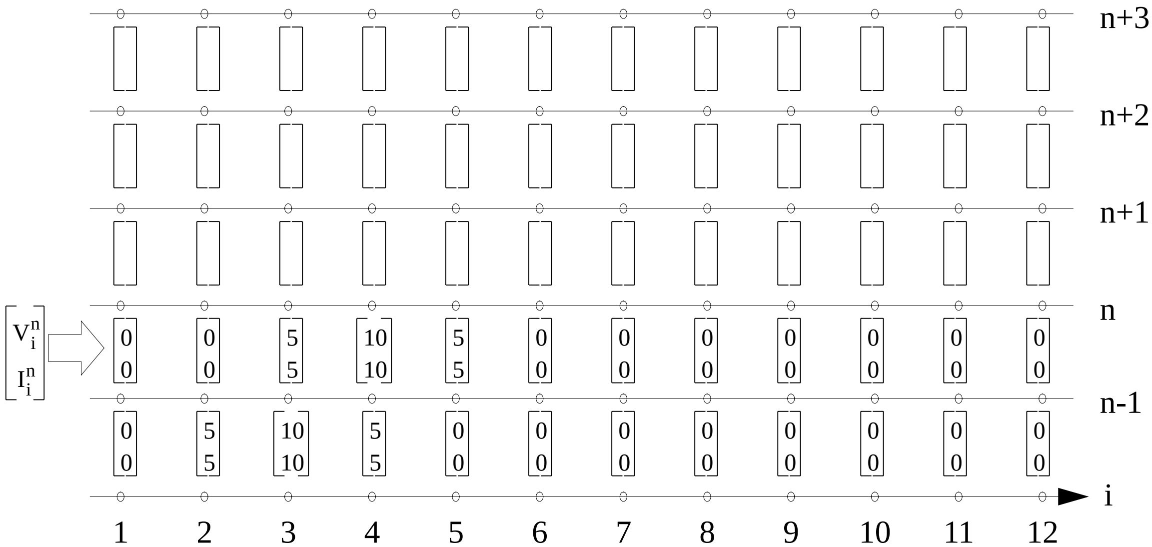
and the second order accurate centered difference approximation:

$$\frac{\partial V}{\partial x} = \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x} + O(\Delta x^2) .$$

(a) Determine and write out the explicit Leap-Frog update equations which approximate the above coupled partial differential equations (do not use half-integer notation).

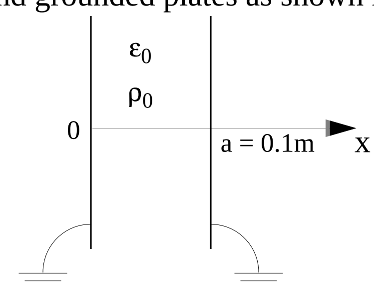
(b) Draw the computational molecule for the resulting scheme using solid dots for the voltage and hollow dots for the current.

(c) Given the value of the solution vector for the two bottom rows of the grid shown in the figure, determine the values of the grid function for the following three time steps (assume the values C = 1 F/m, L = 1 H/m, and $\Delta x = \Delta t = 1$).



(d) Determine the number of independent grids in your scheme, and formulate a new scheme based on this one which contains only one grid (you will require half-integer notation for one of your variables).

15 4) Consider the electric field generated by a space charge with **uniform** density $\rho(x) = \rho_0$ between two infinite parallel and grounded plates as shown in the figure.



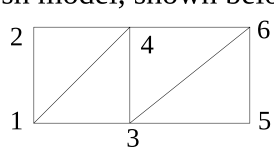
$\varepsilon_0 = 8.854 \times 10^{-12}$ [F/m]

$\rho_0 = 1.0 \times 10^{-6}$ [C/m^3]

(a) Solve the problem analytically by integrating the Poisson equation.

(b) Solve the problem using the finite element method. Discretize the 1D domain into 3 equal elements and assume a linear approximation of the solution. Compare this solution at the nodes to that of part (a).

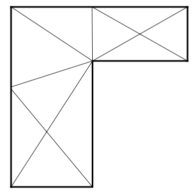
15 5) For the finite element mesh model, shown below,



$V_1 = V_2 = 0$
$V_5 = V_6 = 100$

determine the stiffness matrix [S] before and after introducing the boundary conditions (assume that in each triangle there is one 90^o and two 45^o angles).

15 6) Using the ***Cuthill-McKee*** algorithm, number all nodes in the following mesh (assume the upper right node is the root node).



Determine the structure of the matrix [S], the value of the half-bandwidth and the structure of [S] in banded form (i.e. using 0 - for zero elements and X - for nonzero elements)

15 7) Consider the one-dimensional Poisson's equation boundary value problem given as:

$$\text{ODE:} \quad \frac{d^2}{dx^2}u(x) = xe^x \qquad 0 \le x \le \pi \qquad \text{B.C.'s:} \quad u(0) = 0,\ u(\pi) = 0.$$

(a) Show that the exact solution is given by:

$$u(x) = xe^x - 2e^x + \left(\frac{2e^\pi}{\pi} - e^\pi - \frac{2}{\pi}\right)x + 2 \qquad 0 \le x \le \pi.$$

(b) Use Galerkin's method (i.e. specific instance of the Method of Moments) with basis functions given as:

$$u_n = \sin(nx), \qquad n = 1, 2, 3, \ldots, N$$

to create an approximate expansion of the form:

$$u(x) = \sum_{n=1}^{N} \alpha_n u_n = \underline{u}^T \underline{\alpha} = \underline{\alpha}^T \underline{u}$$

and formulate the matrix equation which approximates the solution to the above problem. Derive the components of the matrix equation for any size expansion (i.e. any N). Find the approximate solution for the case N = 3. Use the inner product defined as

$$(f, g) = \int_0^\pi fg\,dx.$$

You may need the following formulae:

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } m = n \\ 0 & \text{if } m \ne n \end{cases}, \qquad \int xe^x dx = e^x(x-1)$$

and

$$\int xe^x \sin(mx)\,dx =$$

$$\frac{xe^x}{1+m^2}[\sin(mx) - m\cos(mx)] - \frac{e^x}{[1+m^2]^2}[(1-m^2)\sin(mx) - 2m\cos(mx)]$$